The similarity between the quantum searching algorithm and the course of two rigid bodies' collision

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Abstract

By study the attribute of operation of inversion about average in quantum searching algorithm, we find the similarity between the quantum searching and the course of two rigid bodies' collision. Some questions are discussed from this similarity.

The quantum computation is based on qubits, each of which usually is a two-state quantum system. Unlike the classical bit, the qubit can lie in the superposition of the two states. This peculiar attribute leads to the quantum parallel computation, which is the main characteristics of computation [1]. Grover proposed the quantum searching algorithm which can realize fast searching, such as finding an object in unsorted data consisting of N items [2]. This algorithm can be described as following. Initially, the system consisting of n qubits is set in a superposition of all basis states whose amplitudes are the same. Each basis state is corresponding an item in the data, in which one item is marked and to be searched. Through the evolution under the internal interactions and external conditions, the amplitude in the marked state can be close to 1, and the amplitudes in other states are all close to 0. When a measurement is made, the probability of getting this state is close to 1, so that this state is searched [2]. The number of marked items can be lager than 1 by generalizing the algorithm above. In this paper, we will discuss the similarity between the quantum algorithm and course of two rigid bodies' collision and some relative problems.

For a system consisting n qubits, its state can be described by a $N = 2^n$ dimensions vector in Hilbert space. The system start with the initial state

$$|\Psi_0 >= \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i > \tag{1}$$

The states to be searched constitute the marked state, and the other ones constitute the collective state [4]. For convenience, we first consider the simplest case that the number of the marked state is 1, and $N \gg 1$. We define two operators which are denoted as C and D. If C is applied to a superposition of states, it only invert the amplitude in the marked state, and leaves the other states unaltered. The operator D can be defined as $D \equiv 2P - I$, where P is defined as $P \equiv (1/N)\sum_{i,j=0}^{N-1} |i >< j|$. In matrix notion, P is a $N \times N$ matrix whose elements are all equal to $1/N$, and I is a $N \times N$ unit matrix. The complete operation is presented as $U \equiv DC$. Each operation U is called an iteration. After the operation U is repeated $O(\sqrt{N})$ times, the amplitude in the marked state can almost reach 1. The algorithm can be generalized to the case of multiple solutions [3]. We first consider operation of D. If D is applied to a system in state $|\Psi >= \sum_{i=0}^{N-1} c_i|i >$, it transformed the system into state $|\Psi >= \sum_{i=0}^{N-1} c_i|i >$. One can easily see the relationship of the amplitudes c_i and $c_i^{'}$

$$c_i^{'} = 2A - c_i \tag{2}$$

where $A = (1/N)\sum_{i=0}^{N-1} c_i$, which is the average of all amplitudes of the system. Through easy computing, we find that

$$\sum_{i=0}^{N-1} c_i = \sum_{i=0}^{N-1} c_i^{'} \tag{3}$$

and

$$\sum_{i=0}^{N-1} c_i * c_i^* = \sum_{i=0}^{N-1} c_i^{'} * c_i^{'*} \tag{4}$$

where c_i^* denotes the complex conjugation of c_i. These two equations above show that the sum of all amplitudes and all probabilities are conserved. The basis states can be defined as two classes, which are called marked state and collective state. In general, they contain N_1 and N_2 basis states, respectively. We denote the amplitudes in the marked state as $a_0^{'}$ and the ones in collective state as $b_0^{'}$ before D is applied. The two amplitudes are transformed into

a_1 and b_1 after D is applied to the system. under the conditions above, equation(3) and (4) can be simplify to

$$N_1 a_0^{'} + N_2 b_0^{'} = N_1 a_1 + N_2 b_1 \tag{5}$$

$$N_1 a_0^{'2} + N_2 b_0^{'2} = N_1 a_1^2 + N_2 b_1^2 \tag{6}$$

The two equations above remind us that the operation D be analogous to course of two rigid bodies collision, in which their mass are m_1 and m_2, respectively. If the two bodies' velocities are $u_0^{'}$ and $v_0^{'}$ before the collision and $u_1$and v_1 after the collision, we get the two equations from the laws of conservation of energy and momentum

$$m_1 u_0^{'} + m_2 v_0^{'} = m_1 u_1 + m_2 v_1 \tag{7}$$

$$\frac{1}{2} m_1 u_0^{'2} + \frac{1}{2} m_2 v_0^{'2} = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 v_1^2. \tag{8}$$

We find that equations (7) and (8) have the same form as equations (5) and (6).Their solutions must also have the same forms.The solution for equations (7) and (8) are

$$u_1 = \frac{(m_1 - m_2) u_0^{'} + 2 m_2 v_1^{'}}{m_1 + m_2} = 2c_0 - u_0^{'} \tag{9}$$

$$v_1 = \frac{2 m_1 u_0^{'} + (m_2 - m_1) v_0^{'}}{m_1 + m_2} = 2c_0 - v_0^{'} \tag{10}$$

where $v_c = (m_1 u_0^{'} + m_2 v_0^{'})/(m_1 + m_2)$ is the velocity of the mass center of the system. One can easily find that it is conserved during the course of the two bodies' collision.

Now we consider the relationship between the quantum system consisting of N basis states $|i>(i = 0, 1, 2, ..., N-1)$ and a classical system consisting of N unit masses $m_{0i}(i = 0, 1, .., N-1)$. The basis state $|i>$ can be analogous to the unit mass m_{0i}, the amplitude in this state a_i can be analogous to the velocity v_i of this mass. Initially, the all masses have the same velocity, which is analogous to the initial states of quantum system. The N_2 unit masses compose the body m_2, which is analogous to the marked state in quantum algorithm. The other N_1 unit masses compose the body m_1 which analog to the collective state. One easily see that m_1 and m_2 are in fact the masses of the two bodies. The velocity of the mass center of the system v_c is the average of all unit masses, which is analogous to the average of all amplitudes A in the quantum system. The operation C can be seen as the operation which inverts the velocity of mass m_2, and leaves body 2 unaltered. The inversion of velocity of body 2 and the collision of two bodies compose the complete iteration of operation in classical system corresponding to U used in quantum algorithm. By compering equations (5) and (6) with equations (7) and (8), one finds that the operation D can be seen the course of collision of the two masses m_1 and m_2. From equations (9) and (10), one find that in the frame of the mass center, the velocity of two masses are $u_0' - c_0$, and $v_0' - c_0$ before the collision. But after the collision, they are $c_0 - u_0'$, and $c_0 - v_0'$. This fact means that the velocities are inverted by the collision. In the laboratory frame, they are $2c_0 - u_0'$, and $2c_0 - v_0'$. The operation D, which is called inversion about average, transforms the amplitudes of marked state and collective state from a_0' and b_0' to $2A - a_0'$ and $2A - b_0'$. The relationship between operation D and the collision is obvious. We denote the amplitudes of the marked state and collective state are a_n and b_n after n iterations of U. Because the amplitudes can be analog to velocities in the classical system, we can easily get the equation

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \frac{1}{N_1 + N_2} \begin{pmatrix} N_1 - N_2 & -2N_2 \\ 2N_2 & N_1 - N_2 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \tag{11}$$

from the results in classical mechanics

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \frac{1}{m_1 + m_2} \begin{pmatrix} m_1 - m_2 & -2m_2 \\ 2m_2 & m_1 - m_2 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \tag{12}$$

One can get the analytic expressions of the two equations above. They are

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} sin(\frac{2n+1}{\sqrt{N}}) \\ \frac{1}{\sqrt{N-1}} cos(\frac{2n+1}{\sqrt{N}}) \end{pmatrix} \tag{13}$$

for quantum algorithm,and

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} v\sqrt{N} sin(\frac{2n+1}{\sqrt{N}}) \\ v cos(\frac{2n+1}{\sqrt{N}}) \end{pmatrix} \tag{14}$$

for the classical system, with the initial conditions $a_0 = b_0 = 1/\sqrt{N}$, $u_0 = v_0 = v$ under the condition that the marked state consisting of one basis state and $N \gg 1$. During the course of collision, energy is transferred between the two masses so that mass m_2 can achieve the total energy of the system under proper conditions that $n = \pi\sqrt{N}/4$. Correspondingly, for the quantum algorithm, the amplitude in the marked state can reach 1 if the number of repetition of U is $n = \pi\sqrt{N}/4$.

According to the classical analog of quantum searching algorithm, we conceive an experiment to simulate and study the algorithm by a simple classical method. There are two rigid bodies, whose masses are $m_1 = N_1 m_0$ and $m_2 = N_2 m_0$, respectively. They move along a smooth path toward right with the same velocity $u_0 = v_0 = v$ shown by Fig.1a. In order to invert the velocity, we set an obstacle at the right end of the path. The velocity of body 2 is inverted but its modulus remains unaltered after it collides with the obstacle. The two velocities after the collision are $u_0^{'} = v$ and $v_0^{'} = -v$, respectively. This collision is corresponding to the operation C used in Grover algorithm. After the collision with the obstacle, the two bodies collide with each other shown as Fig.1b. Their velocities after this collision are u_1 and v_1 shown by Fig.1c. This collision can be analog to the operation D. These two collisions compose the complete iteration analog to the operation U in quantum algorithm. There are 3 possible cases after the second collision: 1) the two bodies both move toward right, 2) body 1 moves toward left and body 2 toward right, and 3)the two bodies both move toward left. Because we only concern the course of collision, the position is not important for us. We can exchange two bodies' positions or set an obstacle at the left of path, in order to make the iteration mentioned above continue. In this way, after $\pi\sqrt{N}/4$ iterations, the velocity of body 1 is near to 0 and the body 2 get almost all the energy of the system under the condition $m_2 \ll m_1$. This is corresponding to the case that the amplitude in the marked state nearly reach 1 and the quantum system lies in the marked state if a measurement is made. We will discuss some relative questions based on the similarity.

1.Simulate Grover quantum searching algorithm on a classical system. Because each unit mass m_{0i} is analogous to each basis state $|i>$, a system consisting of $2^n m_{0i}$ can be used to simulate a quantum searching consisting of n qubits. For example,if $m_1 = 3m_0$ and $m_2 = m_0$, the classical system can simulate a two qubits system. Only through one iteration, $v_1 = 0$, $v_2 = 2v$, body 2 gets all energy of the system, which is analogous to the probability equal to 1 of getting the marked state in quantum algorithm. This result has been gotten by NMR [5]. For the case of $m_1 = 7m_0$, and $m_2 = m_0$,the following results are gotten:

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 5/2 \end{pmatrix} v_0 \tag{15}$$

,

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 11/4 \end{pmatrix} v_0 \tag{16}$$

. One see that body 2 gets $0.945E_0$ after 2 iterations where E_0 is the total energy. Because mass is continuous, the case of $m_2 = m_0$, and $m_1 = N_1 m_0$ is equivalent to the case of $m_2 = km_0$, and $m_1 = kN_1 m_0$, where k is an integer. One can conclude that the case of $m_1 = 3m_0$, and $m_2 = m_0$ is essentially the case that $N_2 = N/4$ discussed in reference [3],where the number of iterations is 1, and the probability of getting the marked state is 1. If we choose the condition that $m_1/m_2 = p/q$, where $q \neq 1$ the common deviser of p and q is only 1, we can simulate Grover algorithm in the case that the marked state contains multiple basis states, which cannot be realized by NMR [?].

2. The limit of quantum searching. Because the two body exchanges their energy after they collide with each other under the condition of $m_1 = m_2$, neither of the two bodies can get all energy of system. This can be analogous to the case that the the condition $N_1 = N_2$, in which Grover algorithm is invalid. We also find that

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 - 4N_2/N \\ 3 - 4N_2/N \end{pmatrix} \tag{17}$$

considering the initial condition $u_0 = v_0 = v$. The equation (17) means that v_1 is inverted after the first iteration under the condition that $N_2 > N/4$. After the velocity of body 2 is inverted, it moves behind body 1. In this case, body 2 reduces its energy after it collides with body 1. In this case, Grover algorithm is not efficient [6].

3.Qualitatively discuss on the number of iteration. Considering the case that $m_1 \gg m_2$ by setting $m_2 = m_0$, $m_1 = (N-1)m_0$ where $N \gg 1$, $u_0 = v_0 = v$,we get $u_1 = v$, and $v_1 = 3v$ after the first iteration. This means that the velocity of body 2 is improved by $2v$ every iteration, and it is improved by $2nv$ after n iterations. When $n = (\sqrt{N}-1)/2$, body 2 gets almost all the energy of the system, which is $Nm_0v^2/2$. This result is in agreement with the ones mentioned by Grover on '2001 international symposium on quantum information (China).

In conclusion, we discuss the similarity between the course of collision and quantum Grover algorithm based on the two conservation equations in quantum searching. The similarity display the relationship between quantum interference and classical collision and can help one to understand the quantum algorithm clearly and deeply: under proper environment, the amplitude in marked state reaches 1 by the interaction of all basis states. This is perhaps the essential meaning of quantum interference. We believe that the similarity could help to understand and resolve the basic problem in quantum mechanics.

References

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Figure Captions

1. The collision of two rigid bodies.

[Figure 1 about here.]

List of Figures

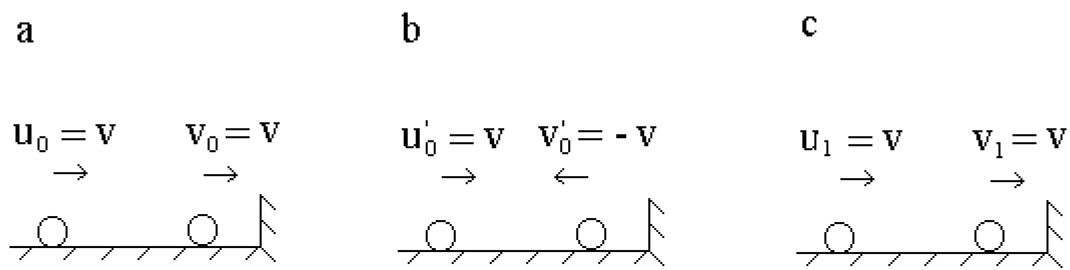

Fig.1

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